A Brief Introduction to Tensors and their properties

1. BASIC PROPERTIES OF TENSORS

1.1 Examples of Tensors

The gradient of a vector field is a good example of a second-order tensor. Visualize a vector field: at every point in space, the field has a vector value $\mathbf{u}(x_1, x_2, x_3)$. Let $\mathbf{G} = \nabla\mathbf{u}$ represent the gradient of $\mathbf{u}$. By definition, $\mathbf{G}$ enables you to calculate the change in **u** when you move from a point **x** in space to a nearby point at $\mathbf{x} + d\mathbf{x}$:

$$d\mathbf{u} = \mathbf{G} \cdot d\mathbf{x}$$

G is a second order tensor. From this example, we see that when you multiply a vector by a tensor, the result is another vector.

This is a general property of all second order tensors. **A tensor is a linear mapping of a vector onto another vector.** Two examples, together with the vectors they operate on, are:

- **The stress tensor**

$$\mathbf{t} = \mathbf{n} \cdot \sigma$$

where **n** is a unit vector normal to a surface, σ is the stress tensor and **t** is the traction vector acting on the surface.

- **The deformation gradient tensor**

$$d\mathbf{w} = \mathbf{F} \cdot d\mathbf{x}$$

where $d\mathbf{x}$ is an infinitesimal line element in an undeformed solid, and $d\mathbf{w}$ is the vector representing the deformed line element.

1.2 Matrix representation of a tensor

To evaluate and manipulate tensors, we express them as **components in a basis**, just as for vectors. We can use the displacement gradient to illustrate how this is done. Let $\mathbf{u}(x_1, x_2, x_3)$ be a vector field, and let $\mathbf{G} = \nabla\mathbf{u}$ represent the gradient of **u.** Recall the definition of **G**

$$d\mathbf{u} = \mathbf{G} \cdot d\mathbf{x}$$

Now, let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis, and express both $d\mathbf{u}$ and $d\mathbf{x}$ as components. Then, calculate the components of $d\mathbf{u}$ in terms of $d\mathbf{x}$ using the usual rules of calculus

$$du_1 = \frac{\partial u_1}{\partial x_1}dx_1 + \frac{\partial u_1}{\partial x_2}dx_2 + \frac{\partial u_1}{\partial x_3}dx_3$$
$$du_2 = \frac{\partial u_2}{\partial x_1}dx_1 + \frac{\partial u_2}{\partial x_2}dx_2 + \frac{\partial u_2}{\partial x_3}dx_3$$
$$du_3 = \frac{\partial u_3}{\partial x_1}dx_1 + \frac{\partial u_3}{\partial x_2}dx_2 + \frac{\partial u_3}{\partial x_3}dx_3$$

We could represent this as a matrix product

$$\begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

Alternatively, using index notation

$$du_i = \frac{\partial u_i}{\partial x_j}dx_j$$

From this example we see that **G** can be represented as a 3×3 matrix. The elements of the matrix are known as the **components of G** in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. All second order tensors can be represented in this form. For example, a general second order tensor **S** could be written as

$$\mathbf{S} \equiv \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

You have probably already seen the matrix representation of stress and strain components in introductory courses.

Since **S** can be represented as a matrix, all operations that can be performed on a 3×3 matrix can also be performed on **S**. Examples include sums and products, the transpose, inverse, and determinant. One can also compute eigenvalues and eigenvectors for tensors, and thus define the log of a tensor, the square root of a tensor, etc. These tensor operations are summarized below.

Note that the numbers $S_{11}, S_{12}, \ldots S_{33}$ depend on the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, just as the components of a vector depend on the basis used to represent the vector. However, just as the magnitude and direction of a vector are independent of the basis, so the properties of a tensor are independent of the basis. That is to say, if **S** is a tensor and **u** is a vector, then the vector

$$\mathbf{v} = \mathbf{S} \cdot \mathbf{u}$$

has the same magnitude and direction, irrespective of the basis used to represent **u**, **v**, and **S**.

1.3 The difference between a matrix and a tensor

If a tensor is a matrix, why is a matrix not the same thing as a tensor? Well, although you can multiply the three components of a vector **u** by any 3×3 matrix,

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

the resulting three numbers (b_1, b_2, b_3) may or may not represent the components of a vector. If they **are** the components of a vector, then the matrix represents the components of a tensor **A**, if not, then the matrix is just an ordinary old matrix.

To check whether (b_1, b_2, b_3) are the components of a vector, you need to check how (b_1, b_2, b_3) change due to a change of basis. That is to say, choose a new basis, calculate the new components of **u** in this basis, and calculate the new matrix in this basis (the new elements of the matrix will depend on how the matrix was defined. The elements may or may not change – if they don't, then the matrix cannot be the components of a tensor). Then, evaluate the matrix product to find a new left hand side, say $(\beta_1, \beta_2, \beta_3)$. If $(\beta_1, \beta_2, \beta_3)$ are related to (b_1, b_2, b_3) by the same transformation that was used to calculate the new components of **u**, then (b_1, b_2, b_3) are the components of a vector, and, therefore, the matrix represents the components of a tensor.

1.4 Formal definition

Tensors are rather more general objects than the preceding discussion suggests. There are various ways to define a tensor formally. One way is the following:

- ***A tensor is a linear vector valued function defined on the set of all vectors***

More specifically, let $S(\mathbf{v})$ denote a tensor operating on a vector. Linearity then requires that, for all vectors $\mathbf{v}, \mathbf{w}$ and scalars α

- $S(\mathbf{v} + \mathbf{w}) = S(\mathbf{v}) + S(\mathbf{w})$
- $S(\alpha \mathbf{v}) = \alpha S(\mathbf{v})$

Alternatively, one can define tensors as sets of numbers that transform in a particular way under a change of coordinate system. In this case we suppose that *n* dimensional space can be parameterized by a set of *n* real numbers x_i. We could change coordinate system by introducing a second set of real numbers $x'_i(x_k)$ which are invertible functions of x_i. Tensors can then be defined as sets of real numbers that transform in a particular way under this change in coordinate system. For example

- A tensor of zeroth rank is a scalar that is independent of the coordinate system.
- A covariant tensor of rank 1 is a vector that transforms as $v'_i = \frac{\partial x_j}{\partial x'_i} v_j$
- A contravariant tensor of rank 1 is a vector that transforms as $v'^i = \frac{\partial x'_i}{\partial x_j} v^j$
- A covariant tensor of rank 2 transforms as $S'_{ij} = \frac{\partial x_i}{\partial x'_k} \frac{\partial x_j}{\partial x'_l} S_{kl}$

- A contravariant tensor of rank 2 transforms as $S'^{kl} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_i}{\partial x_l} S^{ij}$
- A mixed tensor of rank 2 transforms as $S'^i_j = \frac{\partial x'_k}{\partial x_i}\frac{\partial x_j}{\partial x'_l} S^k_l$

Higher rank tensors can be defined in similar ways. In solid and fluid mechanics we nearly always use Cartesian tensors, (i.e. we work with the components of tensors in a Cartesian coordinate system) and this level of generality is not needed (and is rather mysterious). We might occasionally use a curvilinear coordinate system, in which we do express tensors in terms of covariant or contravariant components – this gives some sense of what these quantities mean. But since solid and fluid mechanics live in Euclidean space we don't see some of the subtleties that arise, e.g. in the theory of general relativity.

1.5 Creating a tensor using a dyadic product of two vectors.

Let **a** and **b** be two vectors. The dyadic product of **a** and **b** is a second order tensor **S** denoted by

$$\mathbf{S} = \mathbf{a} \otimes \mathbf{b} \qquad S_{ij} = a_i b_j.$$

with the property

$$\mathbf{S}\cdot\mathbf{u} = (\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{u} = \mathbf{a}(\mathbf{b}\cdot\mathbf{u}) \qquad S_{ij}u_j = (a_i b_k)u_k = a_i(b_k u_k)$$

for all vectors **u**. (Clearly, this maps **u** onto a vector parallel to **a** with magnitude $|\mathbf{a}|\,(\mathbf{b}\cdot\mathbf{u})$)

The components of $\mathbf{a}\otimes\mathbf{b}$ in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

Note that not all tensors can be constructed using a dyadic product of only two vectors (this is because $(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{u}$ always has to be parallel to **a**, and therefore the representation cannot map a vector onto an arbitrary vector). However, if **a**, **b**, and **c** are three independent vectors (i.e. no two of them are parallel) then all tensors can be constructed as a sum of scalar multiples of the nine possible dyadic products of these vectors.

2. OPERATIONS ON SECOND ORDER TENSORS

- **Tensor components**.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis, and let **S** be a second order tensor. The components of **S** in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ may be represented as a matrix

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

where

$$\begin{array}{lll} S_{11} = \mathbf{e}_1\cdot(\mathbf{S}\cdot\mathbf{e}_1), & S_{12} = \mathbf{e}_1\cdot(\mathbf{S}\cdot\mathbf{e}_2), & S_{11} = \mathbf{e}_1\cdot(\mathbf{S}\cdot\mathbf{e}_3), \\ S_{21} = \mathbf{e}_2\cdot(\mathbf{S}\cdot\mathbf{e}_1), & S_{22} = \mathbf{e}_2\cdot(\mathbf{S}\cdot\mathbf{e}_2), & S_{21} = \mathbf{e}_2\cdot(\mathbf{S}\cdot\mathbf{e}_3), \\ S_{31} = \mathbf{e}_3\cdot(\mathbf{S}\cdot\mathbf{e}_1), & S_{32} = \mathbf{e}_3\cdot(\mathbf{S}\cdot\mathbf{e}_2), & S_{31} = \mathbf{e}_3\cdot(\mathbf{S}\cdot\mathbf{e}_3), \end{array}$$

The representation of a tensor in terms of its components can also be expressed in dyadic form as

$$\mathbf{S} = \sum_{j=1}^{3}\sum_{i=1}^{3} S_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$$

This representation is particularly convenient when using polar coordinates, or when using a general non-orthogonal coordinate system.

- **Addition**

Let S and T be two tensors. Then $\mathbf{U} = \mathbf{S} + \mathbf{T}$ is also a tensor.

Denote the Cartesian components of **U, S** and **T** by matrices as defined above. The components of **U** are then related to the components of **S** and **T** by

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} S_{11}+T_{11} & S_{12}+T_{12} & S_{13}+T_{13} \\ S_{21}+T_{21} & S_{22}+T_{22} & S_{23}+T_{23} \\ S_{31}+T_{31} & S_{32}+T_{32} & S_{33}+T_{33} \end{bmatrix}$$

In index notation we would write

$$U_{ij} = S_{ij} + T_{ij}$$

- **Product of a tensor and a vector**

Let $\mathbf{u}$ be a vector and $\mathbf{S}$ a second order tensor. Then

$$\mathbf{v} = \mathbf{S} \cdot \mathbf{u}$$

is a vector.

Let (u_1, u_2, u_3) and (v_1, v_2, v_3) denote the components of vectors $\mathbf{u}$ and $\mathbf{v}$ in a Cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and denote the Cartesian components of $\mathbf{S}$ as described above. Then

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} S_{11}u_1 + S_{12}u_2 + S_{13}u_3 \\ S_{21}u_1 + S_{22}u_2 + S_{23}u_3 \\ S_{31}u_1 + S_{32}u_2 + S_{33}u_3 \end{bmatrix}$$

Alternatively, using index notation

$$v_i = S_{ij} u_j$$

The product

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{S}$$

is also a vector. In component form

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} u_1S_{11} + u_2S_{21} + u_3S_{31} \\ u_1S_{12} + u_2S_{22} + u_3S_{32} \\ u_1S_{13} + u_2S_{23} + u_3S_{33} \end{bmatrix}$$

or

$$v_i = u_j S_{ji}$$

Observe that $\mathbf{u} \cdot \mathbf{S} \neq \mathbf{S} \cdot \mathbf{u}$ (unless $\mathbf{S}$ is symmetric).

Product of two tensors

Let $\mathbf{T}$ and $\mathbf{S}$ be two second order tensors. Then $\mathbf{U} = \mathbf{T} \cdot \mathbf{S}$ is also a tensor.

Denote the components of $\mathbf{U}$, $\mathbf{S}$ and $\mathbf{T}$ by 3×3 matrices. Then,

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$= \begin{bmatrix} T_{11}S_{11} + T_{12}S_{21} + T_{13}S_{31} & T_{11}S_{12} + T_{12}S_{22} + T_{13}S_{32} & T_{11}S_{13} + T_{12}S_{23} + T_{13}S_{33} \\ T_{21}S_{11} + T_{22}S_{21} + T_{23}S_{31} & T_{21}S_{12} + T_{22}S_{22} + T_{23}S_{32} & T_{21}S_{12} + T_{22}S_{22} + T_{23}S_{32} \\ T_{31}S_{11} + T_{32}S_{21} + T_{33}S_{31} & T_{31}S_{12} + T_{32}S_{22} + T_{33}S_{32} & T_{31}S_{13} + T_{32}S_{23} + T_{33}S_{33} \end{bmatrix}$$

Alternatively, using index notation

$$U_{ij} = T_{ik} S_{kj}$$

Note that tensor products, like matrix products, are not commutative; i.e. $\mathbf{T} \cdot \mathbf{S} \neq \mathbf{S} \cdot \mathbf{T}$

Transpose

Let $\mathbf{S}$ be a tensor. The transpose of $\mathbf{S}$ is denoted by $\mathbf{S}^T$ and is defined so that

$$\mathbf{u} \cdot \mathbf{S}^T = \mathbf{S} \cdot \mathbf{u}$$

Denote the components of $\mathbf{S}$ by a 3x3 matrix. The components of $\mathbf{S}^T$ are then

$$\mathbf{S}^T \equiv \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

i.e. the rows and columns of the matrix are switched.

Note that, if $\mathbf{A}$ and $\mathbf{B}$ are two tensors, then

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

Trace

Let $\mathbf{S}$ be a tensor, and denote the components of $\mathbf{S}$ by a 3×3 matrix. The trace of $\mathbf{S}$ is denoted by tr($\mathbf{S}$) or trace($\mathbf{S}$), and can be computed by summing the diagonals of the matrix of components

$$\text{trace}(\mathbf{S}) = S_{11} + S_{22} + S_{33}$$

More formally, let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be any Cartesian basis. Then

$$\text{trace}(\mathbf{S}) = \mathbf{e}_1 \cdot \mathbf{S} \cdot \mathbf{e}_1 + \mathbf{e}_2 \cdot \mathbf{S} \cdot \mathbf{e}_2 + \mathbf{e}_3 \cdot \mathbf{S} \cdot \mathbf{e}_3$$

The trace of a tensor is an example of an *invariant* of the tensor – you get the same value for trace($\mathbf{S}$) whatever basis you use to define the matrix of components of $\mathbf{S}$.

In index notation, the trace is written S_{kk}

- **Contraction**.

Inner Product: Let $\mathbf{S}$ and $\mathbf{T}$ be two second order tensors. The inner product of $\mathbf{S}$ and $\mathbf{T}$ is a scalar, denoted by $\mathbf{S}:\mathbf{T}$. Represent $\mathbf{S}$ and $\mathbf{T}$ by their components in a basis. Then

$$\begin{aligned}\mathbf{S}:\mathbf{T} &= S_{11}T_{11} + S_{12}T_{12} + S_{13}T_{13} \\ &+ S_{21}T_{21} + S_{22}T_{22} + S_{23}T_{23} \\ &+ S_{31}T_{31} + S_{32}T_{32} + S_{33}T_{33}\end{aligned}$$

In index notation $\mathbf{S}:\mathbf{T} \equiv S_{ij}T_{ij}$

Observe that $\mathbf{S}:\mathbf{T} = \mathbf{T}:\mathbf{S}$, and also that $\mathbf{S}:\mathbf{I} = \mathrm{trace}(\mathbf{S})$, where **I** is the identity tensor.

- **Outer product:** Let $\mathbf{S}$ and $\mathbf{T}$ be two second order tensors. The outer product of $\mathbf{S}$ and $\mathbf{T}$ is a scalar, denoted by $\mathbf{S}\cdot\cdot\mathbf{T}$. Represent $\mathbf{S}$ and $\mathbf{T}$ by their components in a basis. Then

$$\begin{aligned}\mathbf{S}\cdot\cdot\mathbf{T} &= S_{11}T_{11} + S_{21}T_{12} + S_{31}T_{13} \\ &+ S_{12}T_{21} + S_{22}T_{22} + S_{32}T_{23} \\ &+ S_{13}T_{31} + S_{23}T_{32} + S_{33}T_{33}\end{aligned}$$

In index notation $\mathbf{S}\cdot\cdot\mathbf{T} \equiv S_{ij}T_{ji}$

Observe that $\mathbf{S}\cdot\cdot\mathbf{T} = \mathbf{S}^T:\mathbf{T}$

- **Determinant**

The determinant of a tensor is defined as the determinant of the matrix of its components in a basis. For a second order tensor

$$\begin{aligned}\det\mathbf{S} &= \det\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33}\end{bmatrix} \\ &= S_{11}(S_{22}S_{33} - S_{23}S_{32}) + S_{12}(S_{23}S_{31} - S_{21}S_{33}) + S_{13}(S_{21}S_{32} - S_{31}S_{22})\end{aligned}$$

In index notation this would read

$$\det(\mathbf{S}) = \tfrac{1}{6} \in_{ijk}\in_{lmn}\, S_{li}S_{mj}S_{nk}$$

Note that if $\mathbf{S}$ and $\mathbf{T}$ are two tensors, then

$$\det(\mathbf{S}) = \det\left(\mathbf{S}^T\right) \qquad \det(\mathbf{S}\cdot\mathbf{T}) = \det(\mathbf{S})\det(\mathbf{T})$$

- **Inverse**

Let $\mathbf{S}$ be a second order tensor. The inverse of $\mathbf{S}$ exists if and only if $\det(\mathbf{S}) \neq 0$, and is defined by

$$\mathbf{S}^{-1}\cdot\mathbf{S} = \mathbf{I}$$

where $\mathbf{S}^{-1}$ denotes the inverse of $\mathbf{S}$ and $\mathbf{I}$ is the identity tensor.

The inverse of a tensor may be computed by calculating the inverse of the matrix of its components. Formally, the inverse of a second order tensor can be written in a simple form using index notation as

$$S^{-1}_{ji} = \frac{1}{2\det(\mathbf{S})} \in_{ipq}\in_{jkl}\, S_{pk}S_{ql}$$

In practice it is usually faster to compute the inverse using methods such as Gaussian elimination.

- **Change of Basis**.

Let $\mathbf{S}$ be a tensor, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Suppose that the components of $\mathbf{S}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are known to be

$$[S^{(\mathbf{e})}] = \begin{bmatrix} S^{(\mathbf{e})}_{11} & S^{(\mathbf{e})}_{12} & S^{(\mathbf{e})}_{13} \\ S^{(\mathbf{e})}_{21} & S^{(\mathbf{e})}_{22} & S^{(\mathbf{e})}_{23} \\ S^{(\mathbf{e})}_{31} & S^{(\mathbf{e})}_{32} & S^{(\mathbf{e})}_{33}\end{bmatrix}$$

Now, suppose that we wish to compute the components of $\mathbf{S}$ in a second Cartesian basis, $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. Denote these components by

$$[S^{(\mathbf{m})}] = \begin{bmatrix} S_{11}^{(\mathbf{m})} & S_{12}^{(\mathbf{m})} & S_{13}^{(\mathbf{m})} \\ S_{21}^{(\mathbf{m})} & S_{22}^{(\mathbf{m})} & S_{23}^{(\mathbf{m})} \\ S_{31}^{(\mathbf{m})} & S_{32}^{(\mathbf{m})} & S_{33}^{(\mathbf{m})} \end{bmatrix}$$

To do so, first compute the components of the transformation matrix [Q]

$$[Q] = \begin{bmatrix} \mathbf{m}_1 \cdot \mathbf{e}_1 & \mathbf{m}_1 \cdot \mathbf{e}_2 & \mathbf{m}_1 \cdot \mathbf{e}_3 \\ \mathbf{m}_2 \cdot \mathbf{e}_1 & \mathbf{m}_2 \cdot \mathbf{e}_2 & \mathbf{m}_2 \cdot \mathbf{e}_3 \\ \mathbf{m}_3 \cdot \mathbf{e}_1 & \mathbf{m}_3 \cdot \mathbf{e}_2 & \mathbf{m}_3 \cdot \mathbf{e}_3 \end{bmatrix}$$

(this is the same matrix you would use to transform vector components from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$). Then,

$$[S^{(\mathbf{m})}] = [Q][S^{(\mathbf{e})}][Q]^T$$

or, written out in full

$$\begin{bmatrix} S_{11}^{(\mathbf{m})} & S_{12}^{(\mathbf{m})} & S_{13}^{(\mathbf{m})} \\ S_{21}^{(\mathbf{m})} & S_{22}^{(\mathbf{m})} & S_{23}^{(\mathbf{m})} \\ S_{31}^{(\mathbf{m})} & S_{32}^{(\mathbf{m})} & S_{33}^{(\mathbf{m})} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1 \cdot \mathbf{e}_1 & \mathbf{m}_1 \cdot \mathbf{e}_2 & \mathbf{m}_1 \cdot \mathbf{e}_3 \\ \mathbf{m}_2 \cdot \mathbf{e}_1 & \mathbf{m}_2 \cdot \mathbf{e}_2 & \mathbf{m}_2 \cdot \mathbf{e}_3 \\ \mathbf{m}_3 \cdot \mathbf{e}_1 & \mathbf{m}_3 \cdot \mathbf{e}_2 & \mathbf{m}_3 \cdot \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} S_{11}^{(\mathbf{e})} & S_{12}^{(\mathbf{e})} & S_{13}^{(\mathbf{e})} \\ S_{21}^{(\mathbf{e})} & S_{22}^{(\mathbf{e})} & S_{23}^{(\mathbf{e})} \\ S_{31}^{(\mathbf{e})} & S_{32}^{(\mathbf{e})} & S_{33}^{(\mathbf{e})} \end{bmatrix} \begin{bmatrix} \mathbf{m}_1 \cdot \mathbf{e}_1 & \mathbf{m}_2 \cdot \mathbf{e}_1 & \mathbf{m}_3 \cdot \mathbf{e}_1 \\ \mathbf{m}_1 \cdot \mathbf{e}_2 & \mathbf{m}_2 \cdot \mathbf{e}_2 & \mathbf{m}_3 \cdot \mathbf{e}_2 \\ \mathbf{m}_1 \cdot \mathbf{e}_3 & \mathbf{m}_2 \cdot \mathbf{e}_3 & \mathbf{m}_3 \cdot \mathbf{e}_3 \end{bmatrix}$$

To prove this result, let **u** and **v** be vectors satisfying

$$\mathbf{v} = \mathbf{S} \cdot \mathbf{u}$$

Denote the components of **u** and **v** in the two bases by $\underline{u}^{(\mathbf{e})}$, $\underline{u}^{(\mathbf{m})}$ and $\underline{v}^{(\mathbf{e})}$, $\underline{v}^{(\mathbf{m})}$, respectively. Recall that the vector components are related by

$$\underline{u}^{(\mathbf{m})} = [Q]\underline{u}^{(\mathbf{e})} \qquad \underline{u}^{(\mathbf{e})} = [Q]^T \underline{u}^{(\mathbf{m})}$$

$$\underline{v}^{(\mathbf{m})} = [Q]\underline{v}^{(\mathbf{e})} \qquad \underline{v}^{(\mathbf{e})} = [Q]^T \underline{v}^{(\mathbf{m})}$$

Now, we could express the tensor-vector product in either basis

$$\underline{v}^{(\mathbf{m})} = \left[S^{(\mathbf{m})}\right] \underline{u}^{(\mathbf{m})} \qquad \underline{v}^{(\mathbf{e})} = \left[S^{(\mathbf{e})}\right] \underline{u}^{(\mathbf{e})}$$

Substitute for $\underline{u}^{(\mathbf{e})}$, $\underline{v}^{(\mathbf{e})}$ from above into the second of these two relations, we see that

$$[Q]^T \underline{v}^{(\mathbf{m})} = \left[S^{(\mathbf{e})}\right] [Q]^T \underline{u}^{(\mathbf{m})}$$

Recall that

$$[Q]\,[Q]^T = [I] \qquad [I]\,\underline{v}^{(\mathbf{m})} = \underline{v}^{(\mathbf{m})}$$

so multiplying both sides by [Q] shows that

$$\underline{v}^{(\mathbf{m})} = [Q] \left[S^{(\mathbf{e})}\right] [Q]^T \underline{u}^{(\mathbf{m})}$$

so, comparing with the first of equation (1)

$$[S^{(\mathbf{m})}] = [Q][S^{(\mathbf{e})}][Q]^T$$

as stated.

In index notation, we would write

$$S_{ij}^{(\mathbf{m})} = Q_{ik} S_{kl}^{(\mathbf{e})} Q_{jl} \qquad Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$$

Another, perhaps cleaner, way to derive this result is to expand the two tensors as the appropriate dyadic products of the basis vectors

$$\begin{aligned} & S_{kl}^{(\mathbf{m})} \mathbf{m}_k \otimes \mathbf{m}_l = S_{kl}^{(\mathbf{e})} \mathbf{e}_k \otimes \mathbf{e}_l \\ & \Rightarrow \mathbf{m}_i \cdot \left[S_{kl}^{(\mathbf{m})} \mathbf{m}_k \otimes \mathbf{m}_l \right] \cdot \mathbf{m}_j = \mathbf{m}_i \cdot S_{kl}^{(\mathbf{e})} \mathbf{e}_k \otimes \mathbf{e}_l \cdot \mathbf{m}_j \\ & \Rightarrow S_{ij}^{(\mathbf{m})} = (\mathbf{m}_i \cdot \mathbf{e}_k) S_{kl}^{(\mathbf{e})} (\mathbf{e}_l \cdot \mathbf{m}_j) \end{aligned}$$

Invariants

Invariants of a tensor are scalar functions of the tensor components which remain constant under a basis change. That is to say, the invariant has the same value when computed in two arbitrary bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. A symmetric second order tensor always has three independent invariants.

Examples of invariants are

1. The three eigenvalues
2. The determinant
3. The trace
4. The inner and outer products

These are not all independent – for example any of 2-4 can be calculated in terms of 1.

In practice, the most commonly used invariants are:

$$I_1 = \text{trace}(\mathbf{S}) = S_{kk}$$
$$I_2 = \tfrac{1}{2}(\text{trace}(\mathbf{S}) - \mathbf{S}\cdot\cdot\mathbf{S}) = \tfrac{1}{2}(S_{ii}S_{jj} - S_{ij}S_{ji})$$
$$I_3 = \det(\mathbf{S}) = \tfrac{1}{6}\in_{ijk}\in_{pqr} S_{ip}S_{jq}S_{kr} = \in_{ijk} S_{i1}S_{j2}S_{k3}$$

Eigenvalues and Eigenvectors (Principal values and direction)

Let **S** be a second order tensor. The scalars λ and unit vectors **m** which satisfy

$$\mathbf{S}\cdot\mathbf{m} = \lambda\mathbf{m}$$

are known as the *eigenvalues* and *eigenvectors* of **S**, or the *principal values* and *principal directions* of **S**. Note that λ may be complex. For a second order tensor in three dimensions, there are generally three values of λ and three unique unit vectors **m** which satisfy this equation. Occasionally, there may be only two or one value of λ. If this is the case, there are infinitely many possible vectors **m** that satisfy the equation. The eigenvalues of a tensor, and the components of the eigenvectors, may be computed by finding the eigenvalues and eigenvectors of the matrix of components.

The eigenvalues of a symmetric tensor are always real, and its eigenvectors are mutually perpendicular (these two results are important and are proved below). The eigenvalues of a skew tensor are always pure imaginary or zero.

The eigenvalues of a second order tensor are computed using the condition $\det(\mathbf{S}-\lambda\mathbf{I}) = 0$. This yields a cubic equation, which can be expressed as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \qquad I_1 = trace(\mathbf{S}),\quad I_2 = (I_1^2 - \mathbf{S}\cdot\cdot\mathbf{S})/2 \quad I_3 = \det(\mathbf{S})$$

There are various ways to solve the resulting cubic equation explicitly – a solution for *symmetric* **S** is given below, but the results for a general tensor are too messy to be given here. The eigenvectors are then computed from the condition $(\mathbf{S}-\lambda\mathbf{I})\mathbf{m} = 0$.

The Cayley-Hamilton Theorem

Let **S** be a second order tensor and let $I_1 = trace(\mathbf{S}),\quad I_2 = (I_1^2 - \mathbf{S}\cdot\cdot\mathbf{S})/2 \quad I_3 = \det(\mathbf{S})$ be the three invariants. Then

$$\mathbf{S}^3 - I_1\mathbf{S}^2 + I_2\mathbf{S} - I_3 = \mathbf{0}$$

(i.e. a tensor satisfies its characteristic equation). There is an obscure trick to show this… Consider the tensor $\mathbf{S}-\alpha\mathbf{I}$ (where α is an arbitrary scalar), and let **T** be the adjoint of $\mathbf{S}-\alpha\mathbf{I}$, (the adjoint is just the inverse multiplied by the determinant) which satisfies

$$\mathbf{T}(\mathbf{S}-\alpha\mathbf{I}) = \det(\mathbf{S}-\alpha\mathbf{I})\mathbf{I} = (-\alpha^3 + I_1\alpha^2 - I_2\alpha + I_3)\mathbf{I}$$

Assume that **T**= $\alpha^2\mathbf{T}_1 + \alpha\mathbf{T}_2 + \mathbf{T}_3$. Substituting in the preceding equation shows that

$$\mathbf{T}_1 = \mathbf{I} \qquad \mathbf{T}_1\mathbf{S} - \mathbf{T}_2 = I_1\mathbf{I} \qquad \mathbf{T}_3 - \mathbf{T}_2\mathbf{S} = I_2\mathbf{I} \qquad \mathbf{T}_3\mathbf{S} = I_3\mathbf{I}$$

Use these to substitute for I_1, I_2, I_3 into

$$\mathbf{S}^3 - I_1\mathbf{S}^2 + I_2\mathbf{S} - I_3 = \mathbf{S}^3 - (\mathbf{S}-\mathbf{T}_2)\mathbf{S}^2 + (\mathbf{T}_3 - \mathbf{T}_2\mathbf{S})\mathbf{S} - \mathbf{T}_3\mathbf{S} = \mathbf{0}$$

3 SPECIAL TENSORS

Identity tensor The identity tensor **I** is the tensor such that, for any tensor **S** or vector **v**

$$\mathbf{I}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{I} = \mathbf{v}$$
$$\mathbf{S}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{S} = \mathbf{S}$$

In any basis, the identity tensor has components

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric Tensor A symmetric tensor $\mathbf{S}$ has the property

$$\mathbf{S} = \mathbf{S}^T$$

The components of a symmetric tensor have the form

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

so that there are only six independent components of the tensor, instead of nine. Symmetric tensors have some nice properties:

- ***The eigenvectors of a symmetric tensor with distinct eigenvalues are orthogonal.*** To see this, let $\mathbf{u}, \mathbf{v}$ be two eigenvectors, with corresponding eigenvalues λ_u, λ_v. Then $\mathbf{v} \cdot [\mathbf{S} \cdot \mathbf{u}] = \mathbf{u} \cdot \left[\mathbf{S}^T \cdot \mathbf{v}\right] = \mathbf{u} \cdot [\mathbf{S} \cdot \mathbf{v}] \Rightarrow \mathbf{v} \cdot \lambda_u \mathbf{u} = \mathbf{u} \cdot \lambda_v \mathbf{v} \Rightarrow (\lambda_u - \lambda_v)\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{u} \cdot \mathbf{v} = 0$.

- ***The eigenvalues of a symmetric tensor are real.*** To see this, suppose that $\lambda, \mathbf{u}$ are a complex eigenvalue/eigenvector pair, and let $\bar{\lambda}, \overline{\mathbf{u}}$ denote their complex conjugates. Then, by definition $\mathbf{S} \cdot \mathbf{u} = \lambda \mathbf{u} \Rightarrow \mathbf{S}\overline{\mathbf{u}} = \bar{\lambda}\overline{\mathbf{u}}$. And hence $\overline{\mathbf{u}} \cdot [\mathbf{S} \cdot \mathbf{u}] = \lambda \overline{\mathbf{u}} \cdot \mathbf{u}$, $\mathbf{u} \cdot [\mathbf{S} \cdot \overline{\mathbf{u}}] = \bar{\lambda} \mathbf{u} \cdot \overline{\mathbf{u}}$. But note that for a symmetric tensor $\overline{\mathbf{u}} \cdot [\mathbf{S} \cdot \mathbf{u}] = \mathbf{u} \cdot \left[\mathbf{S}^T \cdot \overline{\mathbf{u}}\right] = \mathbf{u} \cdot [\mathbf{S} \cdot \overline{\mathbf{u}}]$. Thus $\lambda \overline{\mathbf{u}} \cdot \mathbf{u} = \bar{\lambda} \mathbf{u} \cdot \overline{\mathbf{u}} \Rightarrow \lambda = \bar{\lambda}$.

The eigenvalues of a symmetric tensor can be computed as

$$\lambda_k = \frac{I_1}{3} + 2\sqrt{\frac{-p}{3}} \cos\left\{\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - \frac{2(k-1)\pi}{3}\right\} \qquad k = 1, 2, 3$$

$$p = I_2 - \frac{1}{3}I_1^2 \qquad q = -\frac{2I_1^3 - 9I_1 I_2 + 27 I_3}{27}$$

$$I_1 = trace(\mathbf{S}) \qquad I_2 = \frac{1}{2}\left(I_1^2 - \mathbf{S} : \mathbf{S}\right) \qquad I_3 = \det(\mathbf{S})$$

The eigenvectors can then be found by back-substitution into $[\mathbf{S} - \lambda \mathbf{I}] \cdot \mathbf{m} = \mathbf{0}$. To do this, note that the matrix equation can be written as

$$\begin{bmatrix} S_{11} - \lambda & S_{12} & S_{13} \\ S_{12} & S_{22} - \lambda & S_{23} \\ S_{13} & S_{23} & S_{33} - \lambda \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the determinant of the matrix is zero, we can discard any row in the equation system and take any column over to the right hand side. For example, if the tensor has at least one eigenvector with $m_3 \neq 0$ then the values of m_1, m_2 for this eigenvector can be found by discarding the third row, and writing

$$\begin{bmatrix} S_{11} - \lambda & S_{12} \\ S_{12} & S_{22} - \lambda \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = -m_3 \begin{bmatrix} S_{13} \\ S_{23} \end{bmatrix}$$

- ***Spectral decomposition of a symmetric tensor*** Let $\mathbf{S}$ be a symmetric second order tensor, and let $\{\lambda_i, \mathbf{e}_i\}$ be the three eigenvalues and eigenvectors of **S.** Then $\mathbf{S}$ can be expressed as

$$\mathbf{S} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i$$

To see this, note that $\mathbf{S}$ can always be expanded as a sum of 9 dyadic products of an orthogonal basis. $\mathbf{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. But since $\mathbf{e}_i$ are eigenvectors it follows that $\mathbf{e}_m \cdot (S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \mathbf{e}_k = S_{mk} = \begin{cases} \lambda_k & m = k \\ 0 & m \neq k \end{cases}$

Skew Tensor. A skew tensor $\mathbf{S}$ has the property

$$\mathbf{S}^T = -\mathbf{S}$$

The components of a skew tensor have the form

$$\begin{bmatrix} 0 & S_{12} & S_{13} \\ -S_{12} & 0 & S_{23} \\ -S_{13} & -S_{23} & 0 \end{bmatrix}$$

Every second-order skew tensor has a *dual vector* $\mathbf{w}$ that satisfies

$$\mathbf{S} \cdot \mathbf{u} = \mathbf{w} \times \mathbf{u}$$

for all vectors $\mathbf{u}$. You can see this by noting that $S_{12} = -w_3 \quad S_{13} = w_2 \quad S_{23} = -w_1$ and expanding out the tensor and cross products explicitly. In index notation, we can also write

$$S_{ij} = -\in_{ijk} w_k \qquad w_i = -\frac{1}{2}\in_{ijk} S_{jk}.$$

Orthogonal Tensors An orthogonal tensor $\mathbf{R}$ has the property

$$\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$$

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

An orthogonal tensor must have $\det(\mathbf{S}) = \pm 1$; a tensor with $\det(\mathbf{S}) = +1$ is known as a *proper orthogonal tensor*.

Orthogonal tensors also have some interesting and useful properties:

- Orthogonal tensors map a vector onto another vector with the same length. To see this, let $\mathbf{u}$ be an arbitrary vector. Then, note that $|\mathbf{R}\cdot\mathbf{u}|^2 = [\mathbf{R}\cdot\mathbf{u}]\cdot[\mathbf{R}\cdot\mathbf{u}] = \mathbf{u}\cdot\mathbf{R}^T\mathbf{R}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
- The eigenvalues of an orthogonal tensor are $1, e^{\pm i\theta}$ for some value of θ. To see this, let $\mathbf{u}$ be an eigenvector, with corresponding eigenvalue λ. By definition, $\mathbf{R}\cdot\mathbf{u} = \lambda\mathbf{u}$. Hence, $[\mathbf{R}\cdot\mathbf{u}]\cdot[\mathbf{R}\cdot\mathbf{u}] = \lambda\mathbf{u}\cdot\lambda\mathbf{u} \Rightarrow \mathbf{u}\cdot\mathbf{u} = \lambda^2\mathbf{u}\cdot\mathbf{u} \Rightarrow \lambda^2 = 1$. Similarly, $\lambda\bar{\lambda} = 1$. Since the characteristic equation is cubic, there must be at most three eigenvalues, and at least one eigenvalue must be real.

Proper orthogonal tensors can be visualized physically as rotations. A rotation can also be represented in several other forms besides a proper orthogonal tensor. For example

- The ***Rodriguez representation*** quantifies a rotation as an angle of rotation θ (in radians) about some axis **n** (specified by a unit vector). Given **R**, there are various ways to compute **n** and θ. For example, one way would be find the eigenvalues and the real eigenvector. The real eigenvector (suitably normalized) must correspond to **n;** the complex eigenvalues give $e^{i\theta}$. A faster method is to note that
$$\mathrm{trace}(\mathbf{R}) = 1 + 2\cos\theta \qquad 2\sin\theta\mathbf{n} = \mathrm{dual}(\mathbf{R}-\mathbf{R}^T)$$
- Alternatively, given **n** and θ, **R** can be computed from
$$\mathbf{R} = \cos\theta\mathbf{I} + \mathbf{W}\cdot\mathbf{W}(1-\cos\theta) + \mathbf{W}\sin\theta$$
where **W** is the skew tensor that has **n** as its dual vector, i.e. $W_{ij} = -\in_{ijk} n_k$. In index notation, this formula is
$$R_{ij} = \cos\theta\delta_{ij} + n_i n_j(1-\cos\theta) - \sin\theta \in_{ijk} n_k$$

Another useful result is the **Polar Decomposition Theorem,** which states that invertible second order tensors can be expressed as a product of a symmetric tensor with an orthogonal tensor:
$$\mathbf{A} = \mathbf{R}\cdot\mathbf{U} = \mathbf{V}\cdot\mathbf{R} \qquad \mathbf{R}\mathbf{R}^T = \mathbf{I} \qquad \mathbf{U} = \mathbf{U}^T \quad \mathbf{V} = \mathbf{V}^T$$
Moreover, the tensors $\mathbf{R}, \mathbf{U}, \mathbf{V}$ are unique. To see this, note that

- $\mathbf{A}^T\mathbf{A}$ is symmetric and has positive eigenvalues (to see that it's symmetric, simply take the transpose, and to see that the eigenvalues are positive, note that $d\mathbf{x}\cdot(\mathbf{A}^T\cdot\mathbf{A})\cdot d\mathbf{x} > 0$ for all vectors d**x**).
- Let λ_k^2 and $\mathbf{m}_k$ be the three eigenvalues and eigenvectors of $\mathbf{A}^T\mathbf{A}$. Since the eigenvectors are orthogonal, we can write $\mathbf{A}^T\mathbf{A} = \sum_{k=1}^{3}\lambda_k^2\mathbf{m}_k\otimes\mathbf{m}_k$.
- We can then set $\mathbf{U} = \sum_{k=1}^{3}\lambda_k\mathbf{m}_k\otimes\mathbf{m}_k$ and define $\mathbf{R} = \mathbf{A}\mathbf{U}^{-1}$. **U** is clearly symmetric, and also $\mathbf{U}^2 = \mathbf{A}^T\mathbf{A}$. To see that **R** is orthogonal note that: $\mathbf{R}^T\mathbf{R} = \mathbf{U}^{-T}\mathbf{A}^T\mathbf{A}\mathbf{U}^{-1} = \mathbf{U}^{-T}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I}$.
- Given that **U** and **R** exist we can write $\mathbf{R}\cdot\mathbf{U} = [\mathbf{R}\cdot\mathbf{U}\cdot\mathbf{R}^T]\cdot\mathbf{R}$ so if we define $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$ then $\mathbf{A} = \mathbf{V}\mathbf{R}$. It is easy to show that **V** is symmetric.
- To see that the decomposition is unique, suppose that $\mathbf{A} = \widehat{\mathbf{R}}\widehat{\mathbf{U}}$ for some other tensors $\widehat{\mathbf{R}}, \widehat{\mathbf{U}}$. Then $\mathbf{A}^T\mathbf{A} = \widehat{\mathbf{U}}^2$. But $\mathbf{A}^T\mathbf{A}$ has a unique square root so $\widehat{\mathbf{U}} = \mathbf{U}$. The uniqueness of **R** follows immediately.